THE SCALAR ORBIT OF THE VIRTUAL KÄHLER–RICCI FLOW: SKETCH OF A RESEARCH PROGRAMME

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Abstract. This document collects ideas on an underdeveloped theory that we have called *virtual Kähler–Ricci flow*. These are *pensées* (2014–2026) about a generalization of the Kähler–Ricci flow, meant to represent meromorphic functions in one complex variable by almost geometric structures (almost in the sense of singular/non-singular metrics). We are now aware that our original formulation, with polynomials, might represent meromorphic functions only in a trivial manner; moreover, by a theorem of Ostrowski, functions such as the Riemann zeta function satisfy no algebraic differential equation, which suggests replacing polynomials by series and allowing differential equations of infinite order. Yet these notes might still be useful to future researchers.

1. Towards a definition of the virtual Kähler–Ricci flow

In this work we try to represent a meromorphic function by a univariate family of complex geometric objects. Naïvely, a meromorphic function $f(s)$ may or may not have a meaningful representation by a family of geometric objects $M(s)$ depending on a parameter s (for instance, by a family of Kähler manifolds $M(s)$). Now, if we consider such a geometric representation problem, in general, each element $M(s^*)$ of $M(s)$ may or may not admit a metric $g(s^*)$. When $M(s^*)$ admits a metric, one can define its geometry in terms of tensors. On a Riemannian manifold, the Ricci tensor measures the amount by which the volume of the manifold deviates from that of Euclidean space (Figure 1). The trace of the Ricci tensor is the usual scalar

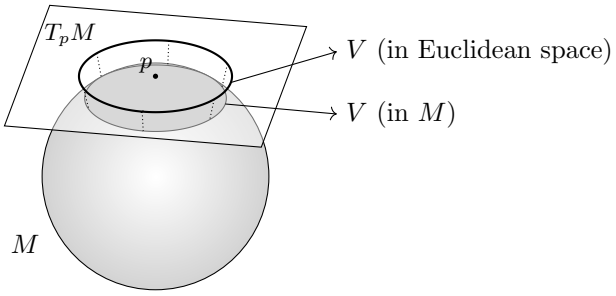


Figure 1. The Ricci tensor measures the amount by which the volume of the manifold deviates from that of Euclidean space.

curvature, say σ. This leads us to the naïve idea of representing some $f(s)$ by $\sigma(s)$.

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We expect there to be values s^* of s for which $M(s^*)$ is not a Riemannian manifold, and $\sigma(s^*)$ would not have a clear interpretation.

As for the complex representation, Kähler manifolds have a particularly well-behaved Ricci tensor. However, when representing such a function, we do not expect $M(s^*)$ to be Kähler for every $s^* \in \mathbb{C}$. This would lead (again, in a naïve manner) to a generalization of the Ricci tensor (or an extension where it is not well defined). This is the origin of the term "virtual": even if the Ricci tensor is not well defined at s^*, we define this virtual ("false") Ricci tensor, by continuing its definition to points where it is not well defined (if necessary, in coordinates).

Now, there is a particular flow describing such a situation, namely, the Ricci flow. In coordinates, the Ricci flow is the tensorial PDE

$$\partial_s g(s) = -2\operatorname{Ric}(s).$$

We do not need the details of how Perelman used the real version of this equation to solve the Poincaré conjecture. The relevant fact is that the Kähler condition is preserved by this equation; hence, if one starts with a Kähler metric, one stays Kähler (see, for instance, [10]). In these inconclusive notes, we consider very general flows. Unlike the Kähler–Ricci flow, these flows are not well behaved.

We now introduce the virtual Kähler–Ricci flow and propose it as a conjectural framework for representing meromorphic functions in one complex variable.

Let (M, g_0) be a compact Kähler manifold of finite dimension n and let Ric_0 be its Ricci curvature tensor. Let P be a formal series (or polynomial) in the variables $g(s)$, $\operatorname{Ric}(s)$ and the covariant derivatives $\nabla^k \operatorname{Ric}(s)$, $k \geq 1$, with coefficients in the field $\mathbb{M}_s$ of meromorphic functions in one complex variable s (note that we do not consider, on purpose, such P to be part of certain rings due to the interdependence between $g(s)$ and $\operatorname{Ric}(s)$; such interdependence could, in principle, be taken into account, using the language of differential algebra, with coefficients in $\mathbb{M}_s$, yet, in this article, we prefer not to do so); in particular, the differential equation below may be of infinite order (cf. Subsection 2.1). We would like to formally generalize the notion of Kähler–Ricci flow as follows.

Definition 1.1. The *virtual Kähler–Ricci flow* $\mathfrak{R}_{P;(M,g_0)}$ of (M, g_0) is

$$\Big\{(M, g(t)) : t \in \mathbb{C},\ \frac{dg}{ds} = P\big(g(s), \operatorname{Ric}(s), \nabla \operatorname{Ric}(s), \dots\big),\ g(0) = g_0\Big\}.$$

By setting $P = -2\operatorname{Ric}(s)$, one obtains the following result due to Bando (see, e.g., [10, Section 1.1.3]).

Theorem 1.2 (Bando)**.** *For real $t \geq 0$ in the maximal interval of existence of the flow, every element $(M, g(t))$ of $\mathfrak{R}_{-2\operatorname{Ric}(s);(M,g_0)}$ is a Kähler manifold.*

Proof. If $P = -2\operatorname{Ric}(s)$, then, for real t, the virtual Kähler–Ricci flow is a Kähler–Ricci flow. But the Kähler condition is preserved under the Kähler–Ricci flow. □

Let $(M, g(t)) \in \mathfrak{R}_{P;(M,g_0)}$. If $g(t)$ is a positive definite Hermitian matrix at each point of M, then $\operatorname{Ric}(t)$ is the Ricci curvature of $g(t)$. This motivates the following generalization.

Definition 1.3. The *virtual Ricci curvature* is $\operatorname{Ric}(t)$ and the *mean virtual scalar curvature* $\sigma(t)$ is the mean of $\operatorname{tr}_{g(t)} \operatorname{Ric}(t)$. The *Ricci orbit* of (M, g_0) is $\operatorname{Ric}(s)$ and its *scalar orbit* is $\sigma(s)$.

The following problems are the motivation for the above generalizations.

Problem 1.4. *Given a meromorphic function $f(s)$, does there exist a Kähler manifold (M, g_0) and a series P (ideally, algebraic, in the sense of differential algebra, and coefficients just in $\mathbb{C}$ rather than in $\mathbb{M}_s$) such that $f(s)$ is the scalar orbit of $\mathfrak{R}_{P;(M,g_0)}$ and, for every zero t of $f(t)$, $g(t)$ is a real, positive definite, symmetric matrix?*

Remark 1.5. Note that, without the positivity condition, the previous problem has a trivial positive answer in case P is a polynomial with non-algebraic coefficients (not $\mathbb{C}$, but $\mathbb{M}_s$), since after a few manipulations, each meromorphic function can be "geometrically" represented in terms of itself. On the other hand, by Ostrowski's theorem (quoted below) the Riemann zeta function $\zeta(s)$ does not satisfy any algebraic differential equation, giving a negative answer in the case of finite order ordinary differential equations in s.

If the answer to the previous problem were yes, then we could associate a Kähler manifold to each zero of $f(s)$.

Problem 1.6. *Given a Kähler manifold (M, g_0), find an algebraic P, or prove that there is none, such that $(M, g(s))$ is Kähler if and only if $\sigma(s) = 0$.*

Consider also the dual problem.

Problem 1.7. *Given a Kähler manifold (M, g_0), find an algebraic P, or prove that there is none, such that $(M, g(s))$ is singular if and only if $\sigma(s) = 0$.*

It would be interesting to restrict this study to series P such that the virtual Kähler–Ricci flow has a unique solution. This would make the notion of orbit in the definitions above a nicer object. A natural question that arises is the following.

Question 1.8. *What is the algebraic structure of the set of algebraic series P which lead to a virtual Kähler–Ricci flow with a unique solution?*

A first test case is $\dim M = 1$. In this case, a good condition for the well-posedness of the flow is that, if $(M, g(0))$ is a Kähler manifold, then $(M, g(t))$ is a Kähler manifold (respectively, singular) for every zero t of σ.

Remark 1.9. Note that taking the mean in Definition 1.3 is necessary: representing meromorphic functions by the scalar curvature itself would not give functions of a single variable, since the scalar curvature also depends on the position on the virtual variety, not only on time.

Remark 1.10. Maybe, in the dual problem, the zeros of the scalar orbit correspond to virtual varieties that are, on average, infinitely curved. These would be singular spaces, and this may give rise to a very particular type of singularities in the Kähler–Ricci curvature. Understanding when one of the varieties of the flow has such a singularity would give us a geometric description of the zeros of the function, and choosing a simple flow that represents the function, we could obtain a geometric advantage over it. When the function cannot be approximated by such a flow, maybe one can build a theory of approximation; note how this results in a cohomology (Section 2).

Remark 1.11. To deal with such spaces, a generalization of the Chern–Gauss–Bonnet theorem (or, more generally, of the Atiyah–Singer theorem) is needed. It

should be more general in the sense that it should cover the case of degenerate manifolds (that is, manifolds which fail to have a real curvature or, in particular, a Hermitian metric). Cf. [2]: their distributions might be interesting for our study, for they give an alternative way of dealing with the cases that arise in our setting; that is, we might be able to avoid metrics. Such a point of view would remove the underlying necessity of using "geometry in coordinates" in order to implement singularities.

2. A cohomology theory for the Riemann Hypothesis

The way we foresee a proof of the Riemann Hypothesis, readable by humans (cf. the recently announced, computer-formalized proof [1], not yet independently verified at the time of writing), is by representing meromorphic functions as solutions to the virtual Kähler–Ricci flow (more precisely, by the mean trace of the generalized Ricci curvature, that is, by the scalar orbit). Maybe this is a cohomology theory, since to each meromorphic function we try to associate the scalar orbit of a manifold-like object. Now the nth cohomology (that is, H_n, built from the scalar orbits of virtual Kähler–Ricci flows on manifolds of dimension n; see the remark below) would correspond to the dimension in which we work.

Remark 2.1. Note that we were trying to represent a meromorphic function exactly by a virtual Kähler–Ricci flow. Yet, an approximation point of view is more pertinent here. The naïve idea is that, the bigger the dimension, the better the approximation, possibly, up to a maximal dimension above which the approximation "would" not get better. The way this "would" work is

$$H_i \longrightarrow H_{i-1},$$

from dimension i to $i-1$, but with a fixed series-like object $\dot{g} = P(g)$ (cf. Ostrowski's Theorem 2.2 on the zeta function below) given by the virtual Kähler–Ricci flow.

The ideas below revolve around three notions: jets (Subsection 2.1), families of varieties (Subsection 2.2) and motivic zeta functions (Subsection 2.3).

2.1. **Jets and equations of infinite order.** To give a first answer to the question of what cohomology is in this context, one should look at de Rham cohomology: the connection ∇ is fundamentally studied by cohomology, and the virtual Ricci flow comes from the chain: metric $g \to \nabla \to$ Ricci curvature $\to$ virtual Ricci flow. Moreover, note that the analogue of a Taylor series, from a naïve model-theoretical point of view (not the real closed, but the algebraically closed part of the theory), is a jet. If we work in the differentiable setting (say real, to obtain a representable picture), then the relevant objects are the derivatives; hence the concept (or "language", in our model-theoretical slang) of jets.

In the case of the virtual Kähler–Ricci flow, however, the derivatives (on the cochains) are not defined: they are rather concepts outside the differentiable category, limit objects not belonging to our initial category. In this direction, one could try to write the virtual flow in the language of the synthetic theory of Ricci curvature bounds [11].

Moreover, the flows should not be expected to be given by finite order PDEs: by the following classical theorem, functions like the ζ function cannot satisfy any finite order algebraic differential equation.

Theorem 2.2 (Ostrowski [7]; see also [5])**.** *The Riemann zeta function $\zeta(s)$ does not satisfy any algebraic differential equation: there is no nonzero polynomial $F \in \mathbb{C}[s, y_0, \dots, y_n]$ such that*

$$F\big(s, \zeta(s), \zeta'(s), \dots, \zeta^{(n)}(s)\big) \equiv 0.$$

This is the main reason why we consider P to be a series in Definition 1.1. In the same direction, maybe the Grothendieck–Katz p-curvature conjecture is related to linear approximations of virtual Kähler–Ricci orbits; this should be studied further, together with Pillay's work [8] on differential algebra and generalizations of Grothendieck's conjecture on the arithmetic of linear differential equations. With this in mind, one would like to define a cohomology theory for the virtual Kähler–Ricci flow, using a spectral sequence to compute the cohomology H^∞. Actually, this is part of a bigger picture: the jets. Gelfand, Kapranov and Zelevinsky go from jets to discriminants [4], which leads to the families of the next subsection. Can we projectivize the whole picture? Does it make sense to make it toric?

2.2. **Families of varieties.** Instead of considering a virtual Kähler–Ricci flow, maybe it would be more interesting to take parametric families of algebraic varieties, vary the parameters, and see when we obtain Kähler varieties; recall that all projective complex algebraic varieties without singular points are Kähler (cf. [6]). In this case one would have to introduce the notion of curvature of an algebraic variety (note that the variety could be singular, i.e., it would not be a curvature in the classical sense), and one would have to see which analytic functions are representable as the trace of this curvature tensor; this would be the algebraic Kähler flow. We would like to associate flat varieties to the zeros of this function (note that a singularity of the metric would destroy this association).

For the Kähler members of such a family, the degeneration of the Hodge–de Rham spectral sequence should be studied and applied to the virtual flow.

More generally, instead of a virtual Kähler–Ricci flow, one may consider a sheaf of PDEs (cf. Grothendieck–Riemann–Roch): the metric is related to the vector bundles (tangent spaces), hence to the cohomology of sheaves ("flow sheaves cohomology"). Along a family, the topological type may change through non-manifold objects (Figure 2).

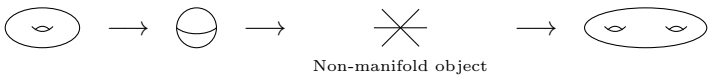


FIGURE 2. Conceptual representation: along a family, the topological type may change through non-manifold objects.

One may see this as a "stack", whose objects are generalized complex varieties which admit a singular metric (groupoid = metrics?). Note that, if the metric is Kähler–Einstein, then $\mathrm{Ric} = \lambda g$, and P becomes a polynomial in g.

2.3. **Motivic zeta functions.** The arithmetic motivation comes from the Bloch–Kato conjecture on values of L-functions: observe how it is related to our point of view on meromorphic functions. Actually, it might be more meaningful to set aside general meromorphic functions and concentrate on L-functions. In the same vein, one could ask the naïve question of whether the Hodge conjecture implies the Riemann Hypothesis. One might use the virtual (Ricci) flows, with motivic measures instead of the usual determinant of g.

From now on, we interchange the roles of s and t.

Let R be a commutative ring and let Sch_R be the category of schemes of finite type over $\operatorname{Spec} R$. Consider maps $\mu\colon \mathrm{Sch}_R \to \tilde{R}$, $X \mapsto \mu(X)$, to a ring $\tilde{R}$, such that

(i) $\mu(X) = \mu(Y) + \mu(X \setminus Y)$ for every closed subscheme Y of X;
(ii) $\mu(X \times Y) = \mu(X)\,\mu(Y)$.

The universal such μ takes values in $K_0(\mathrm{Var}\,/R)$: every μ factors through $K_0(\mathrm{Var}\,/R)$. With such a μ we can define the μ-zeta function of X as

$$\zeta_\mu(X,T) = \sum_{n\ge 0} \mu\big(X^{(n)}\big)\,T^n \in 1 + \tilde{R}[[T]],$$

where $X^{(n)}$ is the nth symmetric product of X.

The plan is to do the same for a family. Instead of a single g, consider a family (g_t, M_t) of complex manifolds with metrics, where t is a complex parameter, or, more generally, a one-parameter family M_t of schemes of finite type over $\operatorname{Spec} R$. For a family of complex varieties, the basic example of μ is the Euler characteristic, $\mu(M_{t_0}) = \chi(M_{t_0})$ for a fixed $t_0 \in \mathbb{C}$. We say that X and X' have the same topological type (with respect to μ) if $\mu(X) = \mu(X')$. A sum $\sum_t \mu(M_t)\,s^t$ over all parameters does not make sense as it stands: each topological type should be counted exactly once, so we need a normalization $\bar{\mu}$ (for instance, if a sphere is counted three times, its contribution $3 \cdot 2$ must be divided by 3). Is the result a motivic measure? This leads to

$$\zeta^*_{M_t}(s) = \sum_{\substack{\text{topological}\\ \text{types}}} \mu(M_t)\,s^{t^*},$$

where $*$ is a way of choosing a representative parameter t^* for each topological type. Alternatively, as for the μ-zeta function above, one may consider $Z_\mu(M_t, s) = \sum_t \mu(M_t)\,s^t$.

Since the Euler characteristic is computed by curvature, one would need a complex version of the Gauss–Bonnet formula

$$\int_{M_t} K_t\,dA + \int_{\partial M_t} k_{g_t}\,d\ell = 2\pi\chi(M_t),$$

where $K_t = K(t)$ is the Gauss curvature and k_g the geodesic curvature.

Example 2.3. Consider a family M_t, $t \in \mathbb{R}$, whose topological type changes as in Figure 3.

M_t							
I	$t \in (-3.5,-2.5]$	$t \in (-2.5,-1.5]$	$t \in (-1.5,-0.5]$	$t \in (-0.5,0.5]$	$t \in (0.5,1.5]$	$t \in (1.5,2.5]$	$t \in (2.5,3.5]$
μ	$\chi = -6$	$\chi = -4$	$\chi = -2$	$\chi = 0$	$\chi = 2$	$\chi = 4$	$\chi = 6$
t^*	$t^*_{-3} = -3$	$t^*_{-2} = -2$	$t^*_{-1} = -1$	$t^*_0 = 0$	$t^*_1 = 1$	$t^*_2 = 2$	$t^*_3 = 3$

FIGURE 3. Conceptual representation: a one-parameter family M_t: for each interval I of parameters with a given topological type, the value $\mu = \chi$ and the representative parameter t^*.

Then, formally,

$$\zeta^*_M(s) = \sum_{n\in\mathbb{Z}} 2n\,s^n.$$

Alternatively, excluding the torus ($\chi = 0$),

$$\zeta(s) = \sum_{n\neq 0}\Big(\frac{2}{2n}\Big)^s = \sum_{n\neq 0}\Big(\frac{2}{\chi(M_{t_n^*})}\Big)^s = \underbrace{\sum_{n\geq 1}\frac{1}{n^s}}_{\text{Riemann}} + \sum_{n\leq -1}\frac{1}{n^s}.$$

We also write, formally, an integral:

$$\zeta_{M_t}(s) = \int \chi(M_t)\, d\mu := \sum_I \int_I \zeta^*_{M_t}(s)\, dt,$$

where I runs over the subsets of $\mathbb{R}$ on which μ is constant. In Example 2.3 it breaks up into sums with constant μ:

$$\zeta_{M_t}(s) = \sum_n \int_{n-1/2}^{n+1/2} 2n\, s^n\, dt.$$

One should express it as an object in $K_0(\mathrm{Var}\,/R)$. This should be the zeta function of the category whose objects are the (g_t, M_t) and whose morphisms are given in terms of the transformation $\dot{g} = P(g, \mathrm{Ric})$, which takes (g_{t_1}, M_{t_1}) to (g_{t_2}, M_{t_2}) via the differential equation.

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