$\vec{I} = (x_1 + L_1^3, \dots, x_n + L_n^3) = (x_1, \dots, x_n) + L$ [The L-graph of a polynomial mapping of the form $x + L$ where $L_1, \dots, L_n$ are cubic forms, 3 type]

$L_i = \sum_{m=1}^{n} c_{im} x_m \qquad c_{im} \in \mathbb{Z}.$

$J_{ij} = \delta_{ij} + 3 L_i^2 \cdot \frac{\partial L_i}{\partial x_j} = \delta_{ij} + 3 L_i^2 c_{ij} = \delta_{ij} + 3 c_{ij} L_i^2.$

[Jacobian conjecture]

$J_{ij} = J_{ji} \Rightarrow c_{ij} L_i^2 = c_{ji} L_j^2 \quad \forall i,j \in [n]$

why should J be symmetric?? [Because, de Bondt, van den Essen, Druzkowski showed that it is enough to prove the Jacobian conjecture for symmetric Jacobian matrix.

$$J = \begin{pmatrix} 1 + 3c_{11}L_1^2 & 3c_{12}L_1^2 & \dots & 3c_{1n}L_1^2 \\ 3c_{21}L_2^2 & 1+3c_{22}L_2^2 & \dots & 3c_{2n}L_2^2 \\ \vdots & & & \vdots \\ 3c_{n1}L_n^2 & 3c_{n2}L_n^2 & \dots & 1+3c_{nn}L_n^2 \end{pmatrix}$$

$\Rightarrow$ there are lots of cancellations. Idea: Graph of cancelations. (CANCELATIONS)

$|J| = K \in \mathbb{K}^* \iff 1 \wedge c_{ij} L_i^2 = c_{ji} L_j^2$ (A)

~~why should the jacobian be symmetric??~~

$$J = \begin{pmatrix} 1+3c_{11}L_1^2 & 3c_{12}L_1^2 & \dots & 3c_{1n}L_1^2 \\ 3c_{12}L_1^2 & 1+3c_{22}L_2^2 & \dots & 3c_{2n}L_2^2 \\ \vdots & & & \\ 3c_{1n}L_1^2 & 3c_{2n}L_n^2 & \dots & 1+3c_{nn}L_n^2 \end{pmatrix}$$

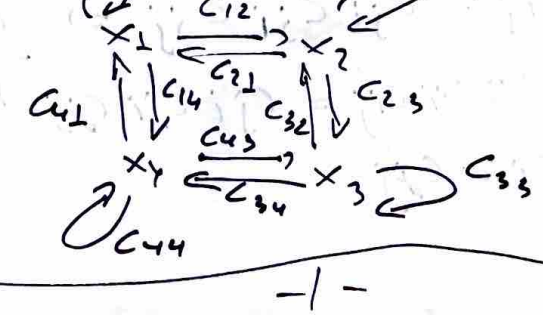
rank $J = n \Rightarrow \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ pairwise disjoint eigenvalues. ~~such that~~ $\forall x \in \mathbb{R}^n$.

Let $\vec{v}_1, \dots \vec{v}_n$ be the eigenvectors

$J = P D P^{-1} = P D P^T$

Ex. $n = 4$

Graph of L



if $c_{ij} = 0$ then there is not such arrow.

c_{ij} can be negative $\in \mathbb{Z}$.

Lemma: $c_{ij} = 0 \iff c_{ji} = 0 \quad \forall i,j$

Proof:

$$c_{ij} L_i^2 = c_{ji} L_j^2 \iff$$

$$\iff c_{ij}\left(\sum_{m=1}^{n} c_{im} x_m\right)^2 = c_{ji}\left(\sum_{m=1}^{n} c_{jm} x_m\right)^2$$

$$c_{ij} = 0 \implies \left(c_{ji} = 0 \vee \sum_{m=1}^{n} \dots c_{jm} x_m \equiv 0\right)$$

$$\iff \left(c_{ji} = 0 \vee \left(c_{j1} = c_{j2} = \dots = c_{jn} = \dots = c_{jn} = 0\right)\right)$$

$$\implies c_{ji} = 0$$

As there is symmetry between i and j — also the other implication

Lemma:

$$\frac{c_{ij}}{c_{ji}} = \frac{c_{jm_1} c_{jm_2}}{c_{im_1} c_{im_2}} \quad \forall m_1, m_2 \in [n]$$

↓ they can be the same

$$\text{or } c_{ji} = \frac{c_{im_2} c_{im_2}}{c_{jm_1} c_{jm_2}} c_{ij} \quad \forall m_1, m_2 \in [n]$$

The following is true

$$c_{ij}\left(c_{im_1} c_{im_2}\right) = c_{ji}\left(c_{jm_1} c_{jm_2}\right) \quad \forall m_1, m_2 \in [n]$$

↓ they are allowed to be the same

Proof: $c_{ij} L_i^2 = c_{ji} L_j^2 \dots$

Corollary: $c_{ij} \neq 0$

$c_{ii} = 1 \quad \forall i \in [n]$ if and only if

$c_{ij} = \pm c_{ji} \quad \forall i \neq j \in [n]$:

Proof:

applying the previous lemma to $m_1 = i$, $m_2 = j$:

$$c_{ij} c_{ii} c_{ij} = c_{ji} c_{ji} c_{jj} \implies c_{ij}^2 = c_{ji}^2$$

$\Leftarrow$:

$$c_{ij} c_{ii} c_{ij} = c_{ji} c_{ji} c_{jj} \iff c_{ij}^2 c_{ii} = c_{ji}^2 c_{jj}$$

$$\iff c_{ii} = c_{jj}$$

~~Def~~: ~~Graph of~~ L-graph of F: $G_L = (V, E)$

~~Def~~ $V = \{x_1, \dots, x_n\}$.

~~$E = \{e_{ij}$~~ $e_{ij} = c_{ij}$ $\quad E = \{e_{ij}\}$.

~~Corollary: If~~

~~Remark:~~

Corollary: If G_L has a connected component L then $J = 1$ and F is regular.

Proof: If G_L has connected comp $L \Rightarrow$ $c_{ij} = 0$ $\forall i \neq j$. $\Rightarrow \deg F = 1$

~~$\Rightarrow$ the~~ but the Jacobian conj. is true for degree 1 polynomials.

?

Remark: We need that all $c_{ii} \neq 0$ $\forall i \in [n]$.

why? Because, otherwise, J is not a constant. Hence $c_{ii} \in \mathbb{Z}^*$ $\forall i \in [n]$

$\Rightarrow$ G_L had all the loops possible. → if the conj is false $\Rightarrow$ counterex. with integer coeffs.

Task: Find interesting cases by looking at necessary condition on G_L for J to be constant.

→ What is the determinant in terms of G_L? Is there another graph G such that the Laplacian of $G = \det J$? (Not quite, as it would mean that G contains both c_{ij} and x_m as edges.) Maybe there is some graph product that gives us the coefficients of $x_{i_1} x_{i_2} \cdots x_{i_2}$ in $\det J$.

Be careful, for this method n must remain fixed.

$$T = (x_1 + (ax_1+bx_2)^3,\ x_2 + (cx_1+dx_2)^3)$$

$$J = \begin{pmatrix} 1+3a(ax_1+bx_2)^2 & 3b(ax_1+bx_2)^2 \\ 3c(cx_1+dx_2)^2 & 1+3d(cx_1+dx_2)^2 \end{pmatrix}$$

$$3b(ax_1+bx_2)^2 = c(cx_1+dx_2)^2$$

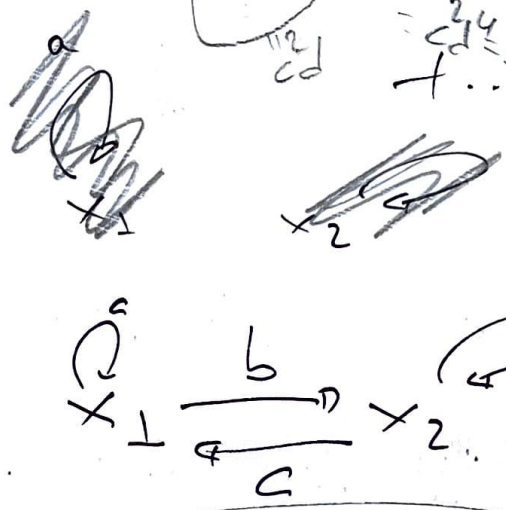
$$\begin{cases} ba^2 = c^3 \\ b^3 = cd^2 \\ bab = ccd \end{cases} \qquad \boxed{\begin{array}{l} ba^2 = c^3 \\ cd^2 = b^3 \\ b^2a = c^2d \end{array}}$$

$$\det J = (1+3a(ax_1+bx_2)^2)(1+3d(cx_1+dx_2)^2) - 3^2b^2(ax_1+bx_2)^4 =$$

$$= 1 + 9ad(ax_1+bx_2)^2(cx_1+dx_2)^2 + 3a(ax_1+bx_2)^2 + 3d(cx_1+dx_1)^2 - \cancel{3b^2}b - 9b^2(ax_1+bx_2)^4$$

$$= 1 + (9ad\,a^2\cdot c^2 - 9b^2a^4)x_1^4 + (9ad\,b^2\cdot d^2 - 9b^6)x_2^4 + \ldots$$

$cd = c^2 \Leftrightarrow b^6$ $\neq 0$ $\forall a,b,c,d$ s.t.



Be careful for this method! It is not suitable for fixed n!

Remark.

$\det J = 1 \Leftrightarrow \forall a,b,c,d$ s.t. ... (struck out)

This is rarely known by people. The truth is that the reason why one has for this form $x_i + L_i^3$

It [illegible] in the case, then my graphs are useless.

$F_i = (x_1 + L_{i1}^3, \dots, x_n + L_{in}^3)$

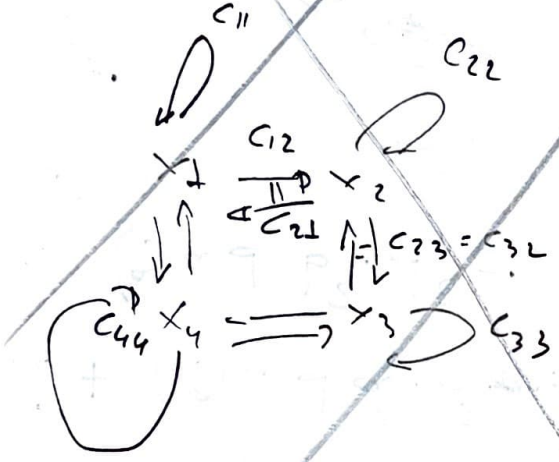
$J_{ij} = \delta_{ij} + 3 L_{ij}^2 \cdot \partial L_{ijj} = D$

$L_{ij} = \sum_{m=1}^{n} c_{ijm} x_m$

$\Rightarrow J_{ij} = \delta_{ij} + 3 \delta_{jmj} c_{ijm} L_{ij}^2 = \delta_{ij} + 3 \delta_{jj} c_{ijj} L_{ij}^2$

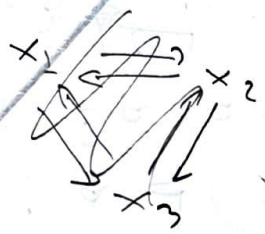
$= \delta_{ij} + 3 c_{ijj} L_{ij}^2$

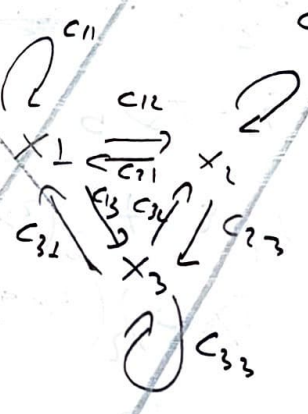
$J_{ij} = J_{ji} \Rightarrow c_{ijj} L_{ij}^2 = c_{jii} L_{ji}^2$



$(a+b+c)(a+b+c) =$

$= a^2 + b^2 + c^2 + 2ab$





$$\begin{cases} L_1 = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ L_2 = c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ L_3 = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{cases}$$

$c_{ij}(c_{i1}x_1 + c_{i2}x_2 + c_{i3}x_3)^2 = c_{ji}(c_{j1}x_1 + c_{j2}x_2 + c_{j3}x_3)^2$

If $c_{ij} = 0 \Rightarrow (c_{ji} = 0 \vee c_{j1} = c_{j2} = c_{j3} = 0)$

$c_{ij} \neq 0$: $c_{ij}(c_{i1}^2 x_1^2 + c_{i2}^2 x_2^2 + c_{i3}^2 x_3^2 + 2c_{i1}c_{i2}x_1x_2$

$+ 2c_{i1}c_{i3}x_1x_3 + 2c_{i2}c_{i3}x_2x_3) = c_{ji}(\dots$

$\Rightarrow$ $c_{ij}c_{i1}^2 = c_{ji}c_{j1}^2$ $\quad c_{ij}c_{i2}^2 = c_{ji}c_{j2}^2$ $\dots$

$c_{ij}c_{i1}c_{i3} = c_{ji}c_{j1}c_{j3}$

$$= \begin{pmatrix} g_1\left(\overline{T}_1(x,y)\right) \\ g_2\left(\overline{T}_2(x,y)\right) \end{pmatrix}$$

$$\begin{cases} \overline{T}_{1x}(x,y)\dot{x} + \overline{T}_{1y}(x,y)\dot{y} = g_1\left(\overline{T}_1(x,y)\right) \\ \overline{T}_{2x}(x,y)\dot{x} + \overline{T}_{2y}(x,y)\dot{y} = g_2\left(\overline{T}_2(x,y)\right) \end{cases}$$

$$\begin{cases} \overline{T}_{1x}(x,y) f_1(x,y) + \overline{T}_{1y}(x,y) f_2(x,y) = g_1\left(\overline{T}_1(x,y)\right) \\ \overline{T}_{2x}(x,y) f_1(x,y) + \overline{T}_{2y}(x,y) f_2(x,y) = g_2\left(\overline{T}_2(x,y)\right) \end{cases}$$

$$\begin{cases} z_x f_1 + z_y f_2 = g_1(z) \\ w_x f_1 + w_y f_2 = g_2(w) \end{cases}$$

1

$\dot{x} = x^2$

$\dot{y} = xy$

$z = x^2$

$w = xy$

$\dot{z} = 2x\dot{x} = 2x^3$

$\dot{w} = \dot{x}y + x\dot{y} = x^2y + x^2y =$

$\frac{d}{dt}(xy) = \frac{\partial(xy)}{\partial x}\dot{x} + \frac{\partial(xy)}{\partial y}\dot{y} = y\dot{x} + x\dot{y} = 2x^2y$

$z = x^2 \quad x = z^{1/2}$

$\dot{z} = 2x\dot{x} = 2xz = 2z^{3/2}$

$\dot{w} = 2x^2y = 2xw = 2z^{1/2}w$

$T_1(x,y) = z$

$T_2(x,y) = w$

$\dot{z} = \frac{\partial T_1}{\partial x}\dot{x} + \frac{\partial T_1}{\partial y}\dot{y} =$

$\dot{w} = \frac{\partial T_2}{\partial x}\dot{x} + \frac{\partial T_2}{\partial y}\dot{y}$

$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} T_{1x} & T_{1y} \\ T_{2x} & T_{2y} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_T \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

$= \begin{Bmatrix} \partial_1(z) \\ \partial_2(w) \end{Bmatrix}$