18th of November of 2022

A note on Dirichlet's L-function, Riemann's ζ function and number π

In 2006, Simon Plouffe found that, for odd k,

$$\pi^5 = \sum_{n=1}^{\infty} \frac{1}{n^5}\left(\frac{a}{e^n-1} + \frac{b}{e^{2n}-1} + \frac{c}{e^{4k}-1}\right) \sum_{n=1}^{\infty} \frac{\nu(n)}{n^5}$$

Now, I know nothing in this series looks to the Riemann's ζ-function!

Def: An α-function, $\alpha(n)$ is a function which makes the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \quad \text{converges for } n > 1.$$

An $\tilde{L}$-function is $\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} =: \tilde{L}(s, \alpha(n))$

Example: a) If $\alpha(n) = 1$, then $\tilde{L}(s, \alpha(n)) = \zeta(s)$.

b) If $\alpha(n) = \left(\frac{a}{e^n-1} + \frac{b}{e^{2n}-1} + \frac{c}{e^{4n}-1}\right)$, then $\tilde{L}(s, \alpha(n)) = \pi^s$

Conjecture: $\frac{\nu(n)!}{4}$

c) If $\alpha(n) = \chi(n)$, then $\tilde{L}(s, \alpha(n)) = \tilde{L}(s, \chi(n)) = L(s, \chi(n))$ a Dirichlet L-series.

where $\chi(n)$ is a Dirichlet character.

Remark: Dirichlet was the one proving $\chi(n) = \chi(n)$ is a Dirichlet character if

$\chi : \mathbb{Z} \to \mathbb{C}$ and

1) $\chi(ab) = \chi(a)\chi(b)$

2) $\chi(a) = \begin{cases} 0 & \text{if } \gcd(a,m) > 1 \\ \neq 0 & \text{if } \gcd(a,m) = 1 \end{cases}$

3) $\chi(a+m) = \chi(a)$, i.e. χ is periodic with period m.

Now, obviously $\nabla(a)$ is not a Dirichlet character. For example, it is not a periodic function.

Question: Is there any ring structure, R, such that ∇ is an R-character?

That is, $\nabla : \mathbb{Z} \to R$ and

1) $\chi(ab) = \chi(a) * \chi(b)$ (* product in R)

2) ...

3) ...

Answer: Well, the periodic property was difficult to deal with, so for now just ignore it. $(R, +_R, *_R)$

Def: An R-arithmetic character $\tilde{\chi}_R$ is a function

$\tilde{\chi}_R : \mathbb{Z} \to R$, $n \mapsto \tilde{\chi}_R(n)$

s.t. 1) $\tilde{\chi}_R(ab) = \tilde{\chi}_R(a) *_R \tilde{\chi}_R(b)$.

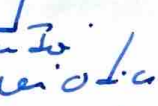
2) $\chi(a) = \begin{cases} 0 & \text{if generalized } \gcd(a,m) > 1 \\ \neq 0 & \text{if generalized } \gcd(a,b) = 1 \end{cases}$

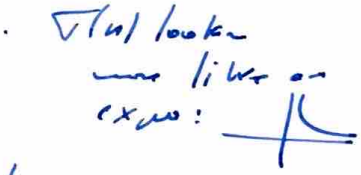
3) $\chi(a+m) = \chi(a) +_R \chi(m)$ and

Remark: we cannot ask that, in the previous def, $\chi(n) = 0_R$, for otherwise it would mean that $\chi(a+n) = \chi(a) +_R \chi(n) = \chi(a) +_R 0_R = \chi(a)$. But $\nabla(n)$ is not periodic, and we would like $\nabla(n)$ to be such a character.

~~Instead, we~~

Question: Is there a good notion of pseudo periodicity?

Answer: Note how the periodicity is needed for convergence reasons. ~~Often~~ we cannot allow $\chi(n)$ to increase with n any how. Maybe $\nabla(n)$ has a pseudo periodicity like 

Finally. $\nabla(n)$ looks more like an $\exp$: 

Good answer: instead of having $\chi(a+n) = \chi(a)$, we could ask that $\chi(a+n) < \chi(a)$. Yes

$\Rightarrow$ Def: (completed) **quasi-character**.

A R-quasi-character $\tilde{\chi}_R$ is

1) $\tilde{\chi}_R : \mathbb{Z} \to \mathbb{C}$

~~$\mathbb{R}$~~ $n \mapsto \tilde{\chi}_R(n)$

s.t

1) $\tilde{\chi}_R(ab) \leq \tilde{\chi}_R(a) *_R \tilde{\chi}_R(b)$

2) $\chi(a) = \begin{cases} 0 \text{ if } \\ \neq 0 \text{ if } \end{cases}$ generalized g.c. $|(a_i)_{i \in \mathbb{N}}|_{\mathbb{R}}$ to $\|=1$

3) $\boxed{\chi(a+m) \leq \chi(a)}$.

Is that is $\chi(m)$ is an infimum.

Because,

$\chi(a+m) = \chi(a) +_R \chi(m)$

$\leq \chi(a) \quad \forall a \in R.$

Hence $\chi(m)$ is an infimum.

Question: What is the generalized g.c.d?

~~Prop: If — Remark conditions for the the canonical def. of χ_R, the $\nabla(a)$ is~~

Task: Find a good Ring (connection with Ring)

Conjecture:

$$\zeta(3) = \pi^3 \cdot \frac{25 \cdot 10^{12}}{644\,258\,754\,165\,467}$$

$$= \frac{1}{25.794350\overline{1666}1868}$$

$$= 23.1279,\ 37\ 517.$$

$$\left\| \left(112x^3 - 1909x^2 + 1077x - 13 \right) \right\| \cdot 584\,303$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \cdots =$$

$$= -\frac{1}{3} \int_0^1 \frac{\ln^3(1-t^2)}{t^3}\, dt =$$

$$= \frac{2}{3} \int_0^\infty \frac{t^2}{1+e^t}\, dt$$

Cont frac: $[0; 25, \overline{1, 3, 1, 6}, 3, 1, \overline{1}, 12, \overline{1, 10}, 3, 2, \overline{1, 19}, \overline{3, 1, 1}]$

But the cont frac of $\zeta(3)/\pi^3$ is

$[0; 25, \overline{1, 3, 1, 6}, 3, \overline{1}, 1, \overline{2, 1}, 10, 3, 2, \overline{1, 19}, \overline{3, 2, 1}]$

The conjecture is false

$$\sum_{n=1}^{\infty} \frac{1}{n^3\left(e^{2\pi n}-1\right)} + \sum_{n=1}^{\infty} \frac{1}{n^3\left(e^{2\pi n}-1\right)\left(e^{2\pi n}+1\right)}$$

The starting point is the Ramanujan's formula.

$$|\coth(\pi n)| = \frac{e^{2\pi n}+1}{e^{2\pi n}-1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{1}{2}\left(\coth(\pi n) - 1\right) =$$

$$= \boxed{\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}\coth(\pi n) - \frac{1}{2}\zeta(3)} =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{i=0}^{\infty}\frac{2^{2i}\,B_{2i}\,(\pi n)^{2i-1}}{(2i)!} - \frac{1}{2}\zeta(3)$$

Express π as a sum/product only depending on n in terms.

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} =$$

$$= \frac{7}{180}\pi^3 - 2\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}\coth(\pi n) - \frac{1}{2}\zeta(3)\right] =$$

$$= \frac{7}{180}\pi^3 - \sum_{n=1}^{\infty}\frac{1}{n^3}\coth(\pi n) + \zeta(3) \Rightarrow$$

$$\zeta(3) = \frac{7}{180}\pi^3 - \sum_{n=1}^{\infty}\frac{1}{n^3}\coth(\pi n) + \zeta(3) \Rightarrow$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^3}\coth(\pi n) = \frac{7}{180}\pi^3}$$

In particular, this series is transcendental. !!!

Also note how good an approximation of $\zeta(3)$ the series in terms $\coth(\pi n)$ converges to 1 very fastly. (exponentially fast)

$$\zeta(3) = \sum_{n=1}^{\infty}\left[\frac{1}{n^3}\left(\frac{28}{e^{2\pi n}-1} - \frac{28}{e^{4\pi n}-1} + \frac{7}{e^{6\pi n}-1}\right)\right] \approx 7.1$$

$$\frac{\pi^3}{2\cdot 3\cdot 7} = \sum_{n=1}^{\infty}\left[\frac{1}{n^3}\left(\frac{720}{e^{2\pi n}-1} - \frac{720}{e^{4\pi n}-1} + \frac{180}{e^{6\pi n}-1}\right)\right] =$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{n^3}\left(\frac{4}{e^{2\pi n}-1} - \frac{4}{e^{4\pi n}-1} + \frac{1}{e^{6\pi n}-1}\right)\right]$$

$$\zeta(3) = \sum_{n=1}^{\infty}\left[\frac{1}{n^3}\left(\frac{28}{e^{2\pi n}-1} - \frac{35}{e^{4\pi n}-1} + \frac{7}{e^{6\pi n}-1}\right)\right] = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} =$$

$$= \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\frac{e^{-2\pi n}}{1-e^{-2\pi n}} = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}e^{-2\pi n k} =$$

$$= \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-2\pi n k}}{n^3} = \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty}\mathrm{Li}_3(e^{-2\pi k})$$

Use relation to Polylogarithm's Li_3 function

$$\frac{7}{180}\pi^3 - 2\zeta(3) = \frac{7}{180}\pi^3 - \frac{\pi^3}{90} = \frac{\pi^3}{180}$$

$$\zeta(3) = \frac{7}{180}\pi^3 - \frac{\pi^3}{180}$$

$$\zeta(3) = \frac{\pi^3}{180}\left(7 - \frac{1}{e^{2\pi}-1}\right)$$

1,2

Splitting the infinite sum Simon Plouffe unearthed a rapidly converging series for zeta(3).

18th of November of 2022

A note on Dirichlet's L-function, Riemann's ζ function and number π

In 2006, Simon Plouffe found that, for odd k,

$$\pi^k = \sum_{n=1}^{\infty} \frac{1}{n^s}\left(\frac{a}{2^n-1} + \frac{b}{2^{2n}-1} + \frac{c}{2^{4k}-1}\right) = \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^s}$$

Now, I show similar to this series leads to the Riemann's ζ function!

Def.: An α-function, $\alpha(n)$ is a function which makes the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} \quad \text{converges for } n > 1.$$

An $\tilde{L}$-function is $\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} =: \tilde{L}(s, \alpha(n))$

Example: a) If $\alpha(n) = 1$, then $\tilde{L}(s, \alpha(n)) = \zeta(s)$.

b) If $\alpha(n) = \left(\frac{a}{2^n-1} + \frac{b}{2^{2n}-1} + \frac{c}{2^{4k}-1}\right)$, then $\tilde{L}(s, \alpha(n)) = \pi^s$

Call it $\frac{\gamma(n)!}{4}$

c) If $\alpha(n) = \chi(n)$, then $\tilde{L}(s, \alpha(n)) = \tilde{L}(s, \chi(n)) =$

$= L(s, \chi(n))$ ← a Dirichlet L-series.

where $\chi(n)$ is a Dirichlet character.

Remark: Dirichlet series has nice property
$\alpha(n) = \chi(n)$ is a Dirichlet character that is

$\chi : \mathbb{Z} \to \mathbb{C}$ and

1) $\chi(ab) = \chi(a)\chi(b)$
2) $\chi(a) = \begin{cases} 0 & \text{if } \gcd(a,m) > 1 \\ \neq 0 & \text{if } \gcd(a,m) = 1 \end{cases}$
3) $\chi(a+m) = \chi(a)$, i.e. χ is periodic with period m.

Now, obviously $\nabla(n)$ is not a Dirichlet character. (For example, it is not a periodic function.)

Question: Is there any ring structure R, such that ∇ is an R-character?

That is, $\nabla : \mathbb{Z} \to R$ and

1) $\chi(ab) = \chi(a) * \chi(b)$
2) ...
3) ...

Answer: Well, the periodic property seems difficult to be dealt with, so for now just ignore it. (R with $+_R, *_R$)

Def: An R-weak-character $\tilde{\chi}_R$ is

$\tilde{\chi}_R : \mathbb{Z} \to R$, $n \mapsto \tilde{\chi}_R(n)$

s.t. 1) $\tilde{\chi}_R(ab) = \tilde{\chi}_R(a) *_R \tilde{\chi}_R(b)$.

2) $\chi(a) = \begin{cases} 0_R & \text{if generalized g.c.d}(a,m) > 1 \\ \neq 0 & \text{if generalized g.c.d}(a,b) = 1 \end{cases}$

3) $\chi(a+m) = \chi(a) +_R \chi(m)$

Remark: We want to note that, in the previous def, $\chi(m) = 0_R$, for otherwise it would mean that $\chi(a+m) = \chi(a) +_R \chi(m) = \chi(a) +_R 0_R = \chi(a)$. But $\nabla(n)$ is not periodic, so we would like $\nabla(n)$ to be such a character.

~~Instead, we~~

Question: Is there a generalization of pseudo-periodicity?

Answer: Note how the periodicity is needed for convergence reasons. ~~We~~ we cannot allow $\chi(n)$ to increase with L any longer. Maybe $\nabla(n)$ has a pseudo-periodicity like ~~this~~ that really. $\nabla(n)$ looks more like an exp: ~~this~~

Good answer: Instead of having $\chi(a+m) = \chi(a)$, we could note that $\chi(a+m) < \chi(a)$. Yeah

⇒ Def: (cont'd) → **quasi-characters**.

An R-quasi-character $\tilde{\chi}_R$ is

1) $\tilde{\chi}_R : \mathbb{Z} \to R$, $n \mapsto \tilde{\chi}_R(n)$

s.t.

1) $\tilde{\chi}_R(ab) \leq \tilde{\chi}_R(a) *_R \tilde{\chi}_R(b)$
2) $\chi(a) = \begin{cases} 0 & \\ \neq 0 & \text{if } - \text{ if generalized g.c.} \end{cases}$ [illegible] to 1 = 1

3) $\boxed{\chi(a+m) \leq \chi(a)}$.

So that is $\chi(AAA)$ is an

infimum.

Because,

$\chi(a+m) = \chi(a) +_R \chi(m)$

$\leq \chi(a) \quad \forall\ a \in \mathcal{U}$.

Here $\chi(m)$ is an infimum.

Question: What is the generalized g.c.d?

~~Prop: If — Remove condition (1) for the the corrected def. of χ_R, then $\nabla(A)$ in~~

Task: Find a good Rij (conflict with Rij)

2.4. *The Local ζ-function; Functional Equation*

In this section $f(\xi)$ will denote a complex valued function defined on k^+; $f(\alpha)$ its restriction to k^*. We let $\mathfrak{z}$ denote the class of all these functions which satisfy the two conditions:

$\mathfrak{z}_1$) $f(\xi)$, and $\hat{f}(\xi)$ continuous, $\in L_1(k^+)$; i.e. $f(\xi) \in \mathfrak{B}_1(k^+)$

$\mathfrak{z}_2$) $f(\alpha)|\alpha|^\sigma$ and $\hat{f}(\alpha)|\alpha|^\sigma \in L_1(k^*)$ for $\sigma > 0$.

A ζ-function of k will be what one might call a multiplicative quasi-Fourier transform of a function $f \in \mathfrak{z}$. Precisely what we mean is stated in

DEFINITION 2.4.1. *Corresponding to each $f \in \mathfrak{z}$, we introduce a function $\zeta(f, c)$ of quasi-characters c, defined for all quasi-characters of exponent greater than 0 by*

$$\zeta(f, c) = \int f(\alpha)c(\alpha)d\alpha,$$

and call such a function a ζ-function of k.

Let us call two quasi-characters equivalent if their quotient is an unrami-

Alpha

https://www.wolframalpha.com/input?i=zeta%283%29%2Fpi%5E3

Continued fraction

Fewer terms | Fraction form

1, 3, 1, 6, 3, 1, 1, 2, 1, 10, 3, 2, 1, 19, 3, 2, 1, 1, 3, 2, 3, 5, 3, 1, 1, 7,
1, 1, 1, 2, 1, 364, 11, 1, 84, 9, 34, 1, 7, 1, 63, 7, 1, 1, 4, 1, 5, 4, 7, 1, 1, 1, 5,
1, 4, 1, 5, 5, 9, 1, 21, 1, 9, 1, 1, 3, 2, 7, 1, 8, 5, 1, 7, 5, 2, 3, 1, 1, 1, 1, 1, 1, 1,
1, 7, 8, 2, 2, 2, 1, 2, 1, 2, 6, 33, 99, 1, 1, 14, 1, 7, 2, 1, 1, 3, 4, 7, 1, 6, 5, 3,
1, 7, 8, 3, 5, 2, 1, 104, 1, 1, 1, 3, 8, 5, 1, 2, 1, 1, 1, 11, 7, 1, 1, 1, 1, 2, 4, 1, 1,
1, 1, 1, 8, 1, 1, 7, 2, 4, 1, 1, 11, 1, 1, 1, 1, 35, 7, 1, 2, 3, 6, 1, 3, 1, 5, 81, 1,
2, 2, 7, 94, 2, 1, 2, 1, 3, 1, 2, 2, 1, 2, 1, 3, 1, 10, 75, 1, 2, 6, 3, 2, 7, 1, 1, 1,
6, 1, 4, 1, 1, 10, 1, 8, 2, 2, 1, 2, 1, 5, 6, 2, 1, 3, 1, 5, 1, 4, 3, 2, 5, 3, 3, 1, 10,
3, 1, 1, 9, 1, 1, 3, 3, 4, 1, 1, 2, 2, 1, 1, 15, 1, 15, 7, 4, 2, 1, 3, 1, 1, 1, 1, 1, 1,
1, 5, 8, 1, 1, 2, 1, 2, 4, 3, 1, 1, 3, 12, 2, 3, 1, 10, 2, 5, 19, 1, 4, 4, 3, 1, 3, 1,
6, 1, 7, 1, 4, 1, 1, 2, 7, 94, 7, 4, 2, 1, 4, 1, 1, 2, 1, 2, 11, 2, 2, 1, 24, 2, 2, 1,
1, 6, 1, 1, 5, 2, 2, 1, 1, 1, 10, 1, 19, 1, 3, 1, 1, 1, 6, 1, 161, 1, 7, 1, 4, 5, 1, 5,

1

100 : 43

200 : 43 + 46 = 89

300 : 129 = 89 + 40

400 : ?

500 : ?

1000 : ?

Numerical conjecture 1

$$\lim_{n\to\infty} \sqrt[n]{\zeta(n)} = 1$$

idea: $B_{2n} \sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}, n\to\infty$ (Stirling formula)

Numerical conjecture 2

Let $P_i(x)$ be the probability of i digit in the continued fraction of x

Conjecture $P_1(\pi) < P_2\left(\left|\frac{\zeta(3)}{\pi^3}\right| < 2\cdot 10^{-6}\right)$

This is not known!

Note:

by a theorem of Levy-Khinchin

$$\sqrt[n]{a_1 a_2 \cdots a_n} \to K_0 \approx 2.685\ldots$$

there are the digits

$$1 = \frac{1}{\sin^2 \frac{\pi}{2}} = \frac{1}{4}\left[\frac{1}{\sin^2 \frac{\pi}{4}} + \frac{1}{\sin^2 \frac{3\pi}{4}}\right]$$

$$= \frac{1}{16}\left[\frac{1}{\sin^2 \frac{\pi}{8}} + \frac{1}{\sin^2 \frac{3\pi}{8}} + \frac{1}{\sin^2 \frac{5\pi}{8}} + \frac{1}{\sin^2 \frac{7\pi}{8}}\right] = \cdots$$

$$= \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}} \tag{3}$$

$$= \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2^{n+1}}}. \tag{4}$$

Taking the termwise limit $n \to \infty$ and using $\lim_{N\to\infty} N \sin(x/N) = x$ for $N = 2^n$ and $x = (2k+1)\pi/2$ yields the series

$$1 = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}, \tag{5}$$

from which (1) follows easily.

Now taking the limit termwise requires some care, as the example $1 = 1/2 + 1/2 = 1/4 + 1/4 + 1/4 + 1/4 = \cdots \to 0 + 0 + 0 \cdots = 0$ shows. In the above case (4) $\to$ (5) it is justified because the kth term in the sum (4) is bounded by $2/(2k+1)^2$ (independently of n) since $\sin x > 2x/\pi$ holds for $0 < x < \pi/2$. ■

The argument in the last step (i.e., interchanging limit and summation) is known as Tannery's Theorem (see [**16**, p. 292], [**5**], or [**4**]); we present it in an appendix at the end of this Note. It is instructive here to check that (and why) the termwise limit (3) $\to$ (5) fails.

Use of Tannery's Theorem can be avoided by the following elementary argument: Sum the inequalities $\sin^{-2} x > x^{-2} > \cot^2 x = \sin^{-2} x - 1$ (which follow from $\sin x < x < \tan x$ for $0 < x < \pi/2$) for $x = (2k+1)\pi/(2N)$ with $k = 0, \ldots, N/2 - 1$. Then (4) implies

$$1 > \frac{8}{\pi^2} \sum_{k=0}^{N/2-1} \frac{1}{(2k+1)^2} > 1 - \frac{1}{N},$$

for $N = 2^n$, and hence (5).

2. RELATED PROOFS. The proof in Section 1 was inspired by two related proofs (# 9 and # 10) among the 14 proofs of Euler's identity (1) collected by Chapman [**6**], and the identity

$$\sum_{k=0}^{N-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2N}} = N^2, \tag{6}$$

which I encountered 25 years ago as a mathematics olympiad problem [**2**]. A proof for (6) for general N is in Section 3. These two related proofs use instead the identities

$$\Pi^k = \sum_{n=1}^{\infty} \frac{1}{n^k} \left(\frac{a}{2^n - 1} + \frac{b}{2^{2n} - 1} + \frac{c}{2^{4n} - 1} \right) =$$

$\frac{1}{A_n}$ → does not depend on k! wonderful!

$$= \sum_{n=1}^{\infty} \frac{1}{n^k} \cdot \frac{1}{A_n}$$

we could do something like:

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \cdot \frac{1}{A_n} = \left[\sum_{n=1}^{\infty} \frac{1}{n^k} \right] * \left[\sum_{n=1}^{\infty} \frac{1}{A_n} \right]$$

some kind of product, like the Hilt product.

Remember that the Hilt product is defined as

$$\left(1 - a t\right)^{-1} * \left(1 - b t\right)^{-1} = \left(1 - ab\, t\right)^{-1}$$

$\Uparrow$

an formal power series

or. $\left(\sum_i [a_i] \right) * \left(\sum_j [b_j] \right) = \sum_{ij} [a_i b_j]$

16.11.22

15.11.22.

According to a math.stackexchange entry, posted on 2014,

$$\zeta(s) = 1 + \cfrac{\frac{1}{2^s}}{1 + \frac{1}{2^s} - \cfrac{\frac{2^s - 1}{3^s}}{1 + \frac{2^s - 1}{3^s} - \cfrac{\frac{3^s - 1}{5^s}}{1 + \frac{3^s - 1}{5^s} - \ddots}}}$$

Let us see what is $\zeta(ns)$. First, note that

$$\frac{p^{ns} - 1}{2^{ns}} = \frac{\left(p^s\right)^n - 1}{\left(2^s\right)^n} = \prod_{\xi_i^n = 1} \frac{p^s - \xi_i}{2^s}$$

$$\Rightarrow \zeta(ns) = \Bigg[1 + \cfrac{\frac{1}{\left(2^s\right)^n}}{1 + \frac{1}{\left(2^s\right)^n} - \cfrac{\prod_{\xi_i^n = 1} \frac{2^s - \xi_i}{3^s}}{1 + \prod_{\xi_i^n = 1} \frac{2^s - \xi_i}{3^s} - \cfrac{\prod_{\xi_i^n = 1} \frac{3^s - \xi_i}{5^s}}{1 + \prod_{\xi_i^n = 1} \frac{3^s - \xi_i}{5^s} - \ddots}}} \Bigg]$$

call this, amplitude

call this $\xi_i^n = 1$ prime-periodic part

Let us now study the prime-periodic part

$$1 + \prod_{\xi_i^n = 1} \frac{p^s - \xi_i}{2^s} = \frac{\prod_{\xi_i^n = 1} 2^s - \prod_{\xi_i^n = 1} \left(p^s - \xi_i\right)}{\prod_{\xi_i^n = 1} 2^s} =$$

$$= 1 + \frac{p^{ns} - 1}{2^{ns}} = \frac{2^{ns} + p^{ns} - 1}{2^{ns}}$$

. Now, by Fermat's last Theorem, it is impossible to find $\alpha \in \mathbb{N}$, s.t. $2^{ns} + p^{ns} = \alpha^{ns}$ for $n \neq 1$ and $s \neq 2$.

use this to prove by induction that $\zeta(3), \zeta(5), \ldots$

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$\zeta(3)$ [20.12.2013]

In our approach, in order to prove that $\zeta(2n+1)$ $n \geq 1$ is a transcendental number, we need to talk about some transcendental number combinatorial number $C_{2n+1} \in \mathbb{Z}_{>0}$ such that

$$\zeta(2n+1) = \kappa_n C_{2n+1} \cdot \pi^{2n+1}$$

$\boxed{\kappa_n \in \mathbb{Q}}$

where C_{2n+1} is a interpolation between

B_{2n} and B_{2n+2}.

Note: What is a combinatorial number? We shall use the following definition:

Def: $C \in \mathbb{Z}_{>0}$ is said to be combinatorial if there is a matroid-like object $\mathcal{M} \in$ [illegible] and some object $m \in \mathcal{M}$ and

$f : \mathcal{M} \to \mathbb{N}$

$f(m) \mapsto C$.

f should have sufficiently nice properties.

E.g. a) $C_n \longmapsto |C_n| = n!$

b) $V_m \longmapsto |V_m|$

Strategy for proving that $\zeta(2n+1)$ [17.12.2013] 10

transcendental:

First observe that $\zeta(2n) = \frac{2^{2n-1}}{(2n)!} |B_{2n}| \pi^{2n} =$

Call $\tilde{C}_{2n} = \frac{2^{2n-1}}{(2n)!} |B_{2n}|$

observe that if $n \gg 1$, then $|B_{2n}| \sim 4\sqrt{\pi \frac{2n}{2}} \left(\frac{2n}{2\pi e}\right)^{2n}$

Hence: $\zeta(2n) \sim \frac{2^{2n-1}}{(2n)!} 4\sqrt{\pi n} \left(\frac{n}{\pi e}\right)^{2n} \pi^{2n}$

Make $\tilde{C}_{2n} \sim \frac{2^{2n-1}}{(2n)!} 4\sqrt{\frac{\pi 2n}{2}} \left(\frac{2n}{2\pi e}\right)^{2n} \frac{1}{\pi^{2n}}$

Now suppose there is a number

$$\tilde{C}_{2n+1} \sim \frac{2^{\frac{2n+1}{2}-1}}{(2n+1)!} 4\sqrt{\frac{\pi(2n+1)}{2}} \left(\frac{2n+1}{2\pi e}\right)^{2n+1} \frac{1}{\pi^{2n+1}}$$

$$\sim C_{2n+1} \sim \frac{2^{\frac{2n+1}{2}-1}}{(2n+1)!} 4\sqrt{\frac{\pi(2n+1)}{2}} \left(\frac{2n+1}{2\pi e}\right)^{2n+1} \frac{1}{\pi^{2n+1}}$$

Suppose that $\tilde{C}_{2n+1} = \frac{2^{\frac{2n+1}{2}-1}}{(2n+1)!} C_{2n+1}$

it is rational $\cdot \sqrt{2}$

Conjecture: $C_{2n+1} = \sqrt{2} \cdot$ a combinatorial number.

It "interpolates" $|B_{2n}|$ and $|B_{2n+2}|$

Note that there is also the following theorem

$$\zeta(n) = \int_{\Omega} \frac{1}{1 - x_1 x_2 \cdots x_n} \, dx_1 dx_2 \cdots dx_n$$

where $\Omega = \prod_{k=1}^{n} (0,1)$ the n hypercube.

$$(0,1) \times (0,1) \times \cdots \times (0,1)$$

So, the combinatorial nature of $\zeta(n)$ for $n \to \infty$ is the limit there is tends.

Sin embargo en la función de Riemann.
El polilogaritmo es trascendental. Y si
$\zeta(2n+1)$ no fuera racional algebraico
en $\mathbb{R}[\pi, \pi^2, \ldots]$?

Pero $\zeta(2n)$ sí porque se "cancela"
la trascendentalidad en $\mathbb{R}[\pi, \pi^2, \ldots]$?

Note:

$$Li_s(1) = \zeta(s)$$

$Li_{2n}(1)$ es algebraico en $\mathbb{R}[\pi, \pi^2, \pi^3, \ldots]$

Conjetura (la "contradicción" con el precio anterior):

$Li_{2n+1}(1)$ es trascendental en $\mathbb{R}[\pi, \pi^2, \pi^3, \ldots]$

Pero $Li_{2n+2}(1) = Li_{2n+2+1}(1)$ "cancela" la trascendentalidad.

instead of 1 : $z \in \mathbb{C}$

where $e_m(n)$ is the mth elementary symmetric function in $\{1, 1/2, \ldots, 1/n\}$. Its specification, for $x = m = 1$, leads us to a strange summation formula

$$\frac{\ln^2 2}{2} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1+\frac{1}{2}+\ldots+\frac{1}{n+1}}{n+2}\right)$$

due to De Doelder [4, Eq. 25].

This method works for most of the hypergeometric summation theorems. With the notation from Slater [7], the generalized hypergeometric series (unilateral and bilateral) read as

$$(0.2a)\qquad {}_{1+p}F_q\left[\begin{matrix}u_0,u_1,\ldots,u_p\\ v_1,\ldots,v_q\end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(u_0)_n\,(u_1)_n\ldots(u_p)_n}{n!\,(v_1)_n\ldots(v_q)_n}z^n,$$

$$(0.2b)\qquad {}_pH_q\left[\begin{matrix}u_1,u_2,\ldots,u_p\\ v_1,v_2,\ldots,v_q\end{matrix};z\right] = \sum_{n=-\infty}^{\infty}\frac{(u_1)_n\,(u_2)_n\ldots(u_p)_n}{(v_1)_n\,(v_2)_n\ldots(v_q)_n}z^n,$$

where the shifted factorial is defined, for a complex number w, by

$$(w)_n = \begin{cases} w(w+1)\ldots(w+n-1), & n=1,2,\ldots,\\ 1, & n=0,\\ (w-1)(w-2)\ldots(w+n), & n=-1,-2,\ldots\end{cases}$$

On the one hand, the hypergeometric series may be expressed in terms of the partial sums of the Riemann zeta function through the symmetric functions generated by finite products (cf. [6])

$$(0.3a)\qquad \prod_{k=1}^{n}(1+x/k) = 1+xH_n+\frac{x^2}{2}(H_n^2-H_n'')+\ldots,$$

$$(0.3b)\qquad \prod_{k=1}^{n}(1-x/k)^{-1} = 1+xH_n+\frac{x^2}{2}(H_n^2+H_n'')+\ldots,$$

$$(0.3c)\qquad \prod_{k=1}^{n}\left(1+\frac{y}{2k-1}\right) = 1+yO_n+\frac{y^2}{2}(O_n^2-O_n'')+\ldots,$$

$$(0.3d)\qquad \prod_{k=1}^{n}\left(1-\frac{y}{2k-1}\right)^{-1} = 1+yO_n+\frac{y^2}{2}(O_n^2+O_n'')+\ldots,$$

whose first two terms

$$(0.4a)\qquad H_n = \sum_{k=1}^{n}\frac{1}{k}, \qquad O_n = \sum_{k=1}^{n}\frac{1}{2k-1},$$

$$(0.4b)\qquad H_n'' = \sum_{k=1}^{n}\frac{1}{k^2}, \qquad O_n'' = \sum_{k=1}^{n}\frac{1}{(2k-1)^2}$$

are closely related to the harmonic numbers.

/

→ Book : Theory of Hypergeometric Function by Aomoto & Kita

Chapter 1
Introduction: the Euler–Gauss Hypergeometric Function

Next suppose that

$$a \geq \sum_{k=1}^{n+1} |a_k| + \sum_{j=1}^{m} |c_j|,$$

and let t lie between 0 and a_{n+1}. Then, by (7), we have

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{j=1}^{m} \cos(c_j x)\right) \cos(tx) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \tag{8}$$

Now integrate (8) with respect to t from 0 to a_{n+1} to get

$$\int_0^\infty \left(\prod_{k=1}^{n+1} \frac{\sin(a_k x)}{x}\right)\left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^{n+1} a_k. \tag{9}$$

Identities (7) and (9) show that if the theorem holds for a pair of integers n, m with $n \geq 1$, $m \geq 0$, then it also holds for the pairs n, $m+1$ and $n+1$, m. Since it holds for $n = 1$, $m = 0$, the proof is completed by induction. □

Remarks 2. Parts of our previous theorems do, of course, overlap with Theorem 4, but this latter theorem does not deal with cases where the identity in (4) fails, whereas the other theorems do. Thus, for example,

$$\int_0^\infty \operatorname{sinc}(x)\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \operatorname{sinc}(x) \operatorname{sinc}\left(\frac{x}{3}\right) dx = \frac{\pi}{2},$$

$$\int_0^\infty \operatorname{sinc}(x) \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{13}\right) dx = \frac{\pi}{2},$$

yet

$$\int_0^\infty \operatorname{sinc}(x) \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi, \tag{10}$$

and this fraction in (10), in accord with Corollary 1, is approximately equal to 0.4999999999992646. When this fact was recently verified by a researcher using a computer algebra package, he concluded that there must be a "bug" in the software. Not so. In the above example, $\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$, but with the addition of $\frac{1}{15}$, the sum exceeds 1 and the identity no longer holds. This is a somewhat cautionary example for too enthusiastically inferring patterns from symbolic or numerical computation.

27.06.2025

A note on Apéry constant $\zeta(3)$

By a Ramanujan formula:

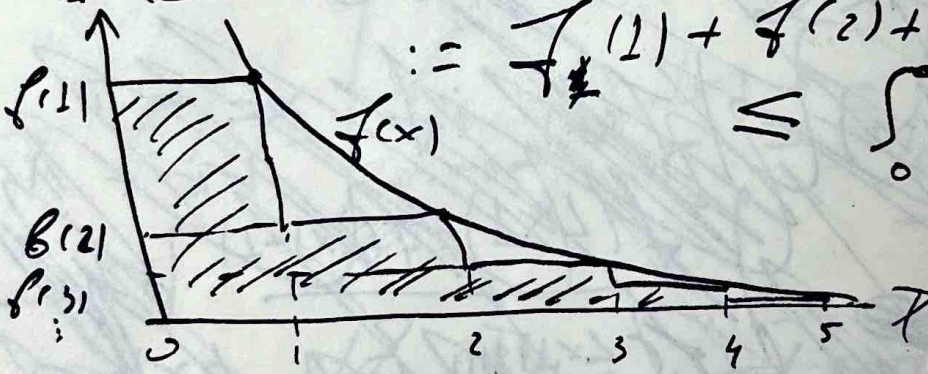
$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty} \frac{1}{k^3(e^{2\pi k}-1)} = 0$$

$\simeq 0,0037\ldots$

$$\frac{\zeta(3)}{\pi^3} + \frac{7}{180} = \frac{\sum_{k=1}^{\infty} \frac{1}{(\pi k)^3(e^{2\pi k}-1)}}{2} := \frac{Z(3)}{2}$$

$$= \frac{1}{\pi^3(e^{2\pi}-1)} + \frac{1}{8\pi^3(e^{4\pi}-1)} + \frac{1}{27\pi^3(e^{6\pi}-1)} + \ldots$$

$$:= f(1) + f(2) + \ldots + f(n) + \ldots \leq \int_0^{\infty} f(x)$$



$$f(1) = \frac{1}{(1\cdot\pi)^3(e^{2\pi\cdot 1}-1)} \quad f(1) = \frac{1}{(2\pi)^3(e^{2\pi\cdot 1}-1)}$$

$$f(k) = \frac{1}{(k\pi)^3(e^{2\pi k}-1)}$$

Let $f(x) = \frac{1}{(x\pi)^3(e^{2\pi x}-1)}$ and

let $\tilde{f}(x) = f(x\sqrt{\pi}) = \frac{1}{x^3(e^{2x}-1)}$

$\Rightarrow f(\pi n) = \tilde{f}(n)$

$$\Rightarrow \zeta(3) \leq \left(\int_1^\infty f(x)\,dx \right) + \frac{1}{\pi^3/(e^{2\pi}-1)}$$

$\int_0^\infty f(x)\,dx$ (**)

$y = \pi x \Rightarrow dy = \pi\,dx \Rightarrow dx = \frac{dy}{\pi}$

$$\Rightarrow \zeta(3) \leq \int_0^\infty f(x)\,dx = \frac{1}{\pi} \int_0^\infty f(y)\,dy$$

Now let $g_1, g_2, g_3, \ldots$ be functions defined as follows:

$g_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$g_i(x) = f(x)$ if $x \in [i, i+1]$

$i \in \mathbb{N}$.

$= 0$ otherwise.

$\lim_{n \to \infty} g_n(x) = f(x)$

$\int_0^\infty g_n(x)\,dx = \sum_{i=1}^{n} \int_i^{i+1} f(x)\,dx$

$= \zeta(3)$!!

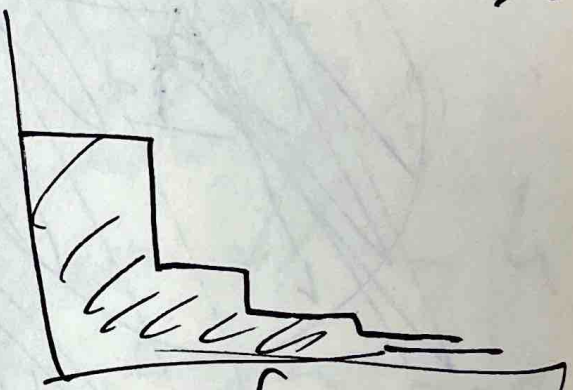
(**) $\int_0^\infty f(x)\,dx$ might be diverging even though $\int_1^\infty f(x)\,dx$ + the first portion (i.e. $\frac{1}{\pi^3(e^{2\pi}-1)}$) converges.

$g_0(x) = f(x)$

$g_i : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$

$$g_i(x) = \begin{cases} g_i(x) = \dfrac{g_{i-1}(\lfloor x \rfloor + 1) + g_{i-1}(x)}{2} \\ \text{if } x \in \mathbb{Z}_{\geq 0} \\ g_i(x) = g_{i-1}(x) \text{ if } x \in \mathbb{Z}_{\geq 0} \end{cases}$$

$\Rightarrow \lim_{n \to \infty} g_n(x) =$?



$$\Rightarrow Z(s) = \int_0^\infty \lim_{n \to \infty} g_n(x) = \int g_\infty(x)$$

$\Rightarrow$ use the Fourier Transform to work it

$$\lim_{n \to \infty} g_n(x)$$

14.11.22

Note on $\zeta(3)$

We saw, in a numerical experiment that

$$\zeta(3) \approx \frac{\pi^3}{0.25\ldots}$$

The denominator does not seem to be rational.

A problem in its own conception and both Maccarthy and Wolfram-alpha are of little numerical use.

Define: $$\boxed{\frac{\zeta(n)}{\pi^n} = \beta_n}$$

Note that $$\frac{\zeta(2n)}{\pi^{2n}} = \frac{(-1)^{n+1} \cdot 2^{2n} \cdot B_{2n}}{2 \cdot (2n)!}$$

$B_{2n+1} = 0$ so we need to interpolate between B_{2n} and B_{2n+2}

In the continued fraction of β_3 the probability to encounter a 1 seem higher than the probability to encounter any other number.

Also: look at big number in the continued fraction

? no, lots of terms 2^k and c^2. Are the infinitely many of them?

14.11.22

A slightly different point of view for $\zeta(s)$.

Def: $\zeta_{\mathbb{N}}(s) = \sum_{n \in \mathbb{N}} n^{-s}$

$$\zeta_{\sqrt{\mathbb{N}^3}}(3) = \sum_{n \in \sqrt{\mathbb{N}^3}} n^{-s} = \zeta(3)$$

$$\zeta(2) = \zeta_{\sqrt{\mathbb{N}^2}}(2) = \sum_{n \in \sqrt{\mathbb{N}^2}} \frac{1}{n^2} =$$

$$= \sum_{n \in \mathbb{N}} \frac{1}{n^2}$$

Example: $\zeta_{\sqrt{\mathbb{N}^4}}(2) = \zeta_{\mathbb{N}^2}(2) = \zeta(4)$

$\mathbb{N}$: $n \in$ . . . . . .

$\mathbb{N}^2$: $\tilde{n} \in$ • • • • •

$\hookrightarrow \zeta(4) = \sum_{n \in \mathbb{N}} \frac{1}{n^4} = \sum_{\tilde{n} \in \mathbb{N}^2} \frac{1}{\tilde{n}^2} =$

$= \sum_{n \text{ is a perfect square}} \frac{1}{n^2} - \sum_{\substack{n \text{ is not} \\ \text{a perfect} \\ \text{sq. n}}} \frac{1}{n^4} =$

$= \zeta(2) - \tilde{\zeta}(4) =$

$= \frac{\pi^2}{6} - \tilde{\zeta}(4) = \frac{\pi^4}{\square}$

$$\boxed{\tilde{\zeta}(4) = \zeta(2) + \zeta(4)}$$

$$\boxed{\tilde{\zeta}(2n+2) := \zeta(2n) - \zeta(2n+2)}$$

↳ Study instead these ζ-functions

$= \alpha \pi^{2n} - \beta \pi^{2n} \pi^2$

Now: take

$$\check{\zeta}(2n+1) := \zeta(2n+1) - \zeta(2n+1)$$

↓

~~Some of the summands of $\zeta(2n+1)$ cancel~~

Def: $\hat{\in}$:

Given a formal series $S = \sum_{i \in I} a_i$, we say that $a_i \hat{\in} S$.

Lemma: If $a \hat{\in} \zeta(n)$ then $a \hat{\in} \zeta(m)$ $\forall m \geq n$.

$\in \mathbb{Z}$, $\in \mathbb{Z}$.

$$\check{\zeta}(2n+1) = \alpha \Pi^{2n} - \beta \overline{\Pi}^{2n} \cdot \Pi =$$

$$= \Pi^{2n} (\alpha - \Pi \beta)$$

Now, as the difference between the elements $a_i \in \mathcal{S}(2n)$ and $b_i \in \mathcal{S}(2n+2k)$ are perfect hypercubes, we can easily use Fermat's Last Thm to get the reduced transcendence of $\mathcal{S}(2n+2k)$ from that of $\mathcal{S}(2)$

↓

Ex: $x^2+y^2=z^2$

↓

$x^4+y^4=z^4$

etc.

But it is not so easy to get it for non multiples of 2?

Homework: Rigorously relate the terms of $\mathcal{S}(3)$ to the non-trivial sols of $x^3+y^3=z^3$